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TECHNICAL NOTES

MHD forced and free convection boundary layer flow near the leading edge

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1. INTRODUCTION

WE ARE frequently concerned with natural convection flow and the consequent heat transfer that arises over surfaces which are at a temperature different from that of the ambient medium. In the case of a heated body cooling in an extensive isothermal medium, the fluid flows adjacent to the hot surface of the body, and this heated fluid eventually rises above the body as a buoyant flow or wake. Similarly, a body colder than the ambient fluid would cause a flow opposite to that due to the heated body, since the fluid adjacent to the body becomes colder and hence heavier than the ambient fluid, resulting in a flow in the direction of the gravitational force. In nature too, many natural or free convection flows occur adjacent to heated or cooled surfaces. Free convection can have a significant effect on forced flows over solid bodies, too. It can alter the flow field and hence the heat transfer rate and wall-shear distribution. The simplest physical model is two-dimensional, mixed forced and free convection along a flat plate. Recent examples of application of this model can be found in the areas of reactor safety, combustion flames, and solar collectors, as well as building energy conservation.

Extensive studies [1–9] have been conducted on mixed convection along vertical, horizontal, or inclined surfaces. It has been generally recognized that $\xi(=Gr/Re^2)$, where Gr is the Grashof number and Re the Reynolds number, is the governing parameter for a vertical plate. Forced convection exists as a limit when ξ goes to zero which occurs at the leading edge, and the free-convection limit can be reached if ξ becomes large. Perturbation solutions have been developed for both limits, since both forced convection and free convection have similarity solutions. Empirical patching of two perturbation solutions has also been carried out to provide a solution by Rajn *et al.* [10] which covers the whole range of ξ. They obtained a finite difference solution applying an algebraic transformation $z = 1(1+\xi^2)$. For a horizontal plate, the axial pressure gradient induced by buoyancy force is $O(Gr/Re^{5/2})$. Numerous solutions have been developed by considering the free-convection effect as a perturbation quantity. Again, forced convection exists as a limit for small ξ and the free convection can be reached as ξ approaches infinitely. Recently, Tingwei *et al.* [11], have studied the effect of forced and free convection along a vertical flat plate with uniform heat flux considering that the buoyancy parameter ξ to be small.

Effects of transversely applied magnetic field on free convection of an electrically conducting fluid past a semi-infinite plate were studied by many researchers [12–15], because of its application in nuclear engineering in connection with the cooling of reactors. In the present note, therefore, we propose to investigate the combined forced and free convection of an electrically conducting fluid past a vertical flat plate at whose surface, the heat flux is uniform and a magnetic field is applied parallel to the direction normal to the plate and is allowed to pass it along with the fluid. The equations governing the flow are developed in Section 2 and are solved numerically using the method of superposition for small values of ξ, the buoyancy parameter.

2. FORMULATION OF THE PROBLEM

Consider the free convection flow of an electrically conducting and viscous incompressible fluid up a heated semi-infinite flat plate extended vertically in the upward direction. Let the temperature of the free-stream be T_0 having the velocity U_0 which is uniform. A magnetic field of strength $B(x)$ is considered to be applied parallel to the y-axis which is normal to the plate and is allowed to move past the plate with fluid. Here we assume that the induced magnetic field produced by the motion of the electrically conducting fluid is negligible. This assumption is valid for smaller magnetic Reynolds number. Further, since no external electric field is applied and the effect of polarization of the ionized fluid is negligible, we may also assume that the electric field $E = 0$. Under the above assumption the boundary layer equations governing the flow past a plate at whose surface the heat flux q is uniform, are (Cobble [14])

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta(T-T_0)-\sigma_0\frac{B^2(x)}{\rho}(U_0-u)+v\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = k\frac{\partial^2 T}{\partial y^2} \quad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (3)$$

subject to the boundary conditions

$$u = v = 0, \quad -k\left(\frac{\partial T}{\partial y}\right) = q \quad \text{at } y = 0$$
$$u \to U_0, \quad T \to T_0 \quad \text{as } y \to \infty. \quad (4)$$

Here (u, v) are velocity components associated with the direction of increasing coordinates (x, y) measured along and normal to the plate, respectively. T is the temperature of the fluid in the boundary layer, g the acceleration due to gravity, β the coefficient of thermal expansion, k the thermal conductivity, σ_0 the electric conductivity, v the kinematic viscosity of the fluid and ρ the reference density of the surrounding fluid.

In formulating equations (1)–(3) it has been assumed that (i) the ratio of thermal diffusivity to magnetic diffusivity is small compared to unity, (ii) fluid property variations are

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limited to density variation which is taken into account only in so far as it effects the buoyancy terms only, (iii) the short circuit assumption applies and (iv) the viscous and electrical dissipation effects are neglected.

Now to reduce equations (1)–(3) to ordinary differential equations, we are to introduce the stream function ψ throughout equations (1)–(3), which is defined in relation (5). These yield the expressions to be transformed, equations (6) and (7):

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{5}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = g\beta(T-T_0) + \frac{\sigma_0 B^2}{\rho}\left(U_0 - \frac{\partial \psi}{\partial y}\right) + v\frac{\partial^3 \psi}{\partial y^3} \tag{6}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = k\frac{\partial^2 T}{\partial y^2}. \tag{7}$$

We now introduce the following set of transformation of Tingwei *et al.* [11] for the dependent and the independent variables:

$$\left.\begin{aligned} &\eta = (U_0/vx)^{1/2}y, && \psi(x,y) = (vU_0x)^{1/2}f(\xi,\eta) \\ &T-T_0 = \left(\frac{U_0 x}{v}\right)^{1/2}\left(\frac{qx}{k}\right)\theta(\xi,\eta), && B(x) = B_0x^{1/2} \\ \text{where}\quad &\xi = Gr/Re^{5/2} && \\ \text{with}\quad &Gr = g\beta q x^4/kv^2, && Re = U_0x/v \end{aligned}\right\} \tag{8}$$

into equations (6) and (7) as well as in boundary conditions (4), we find

$$f''' + \frac{1}{2}ff'' = \xi\left[\frac{3}{2}f'\frac{\partial f'}{\partial \xi} - \frac{3}{2}f''\frac{\partial f}{\partial \xi} - M(f'-1) - \theta\right] \tag{9}$$

$$\theta'' + \frac{1}{2}\sigma(f\theta' - f'\theta) = \frac{3}{2}\sigma\xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right) \tag{10}$$

$$\left.\begin{aligned} f(\xi,0) = f'(\xi,0) = 0, \quad \theta'(\xi,0) = -1 \\ f'(\xi,\infty) = 1, \quad \theta(\xi,0) = 0. \end{aligned}\right\} \tag{11}$$

Here primes denote differentiations of the functions with respect to η only, $\sigma(=v/k)$ the Prandtl number and $M[=(\sigma_0 B_0^3 k/\rho g\beta q)(U_0^3 v)^{1/2}]$ is the magnetic field parameter.

In the present problem the buoyancy force is proportional to $\xi(=Gr/Re^{5/2})$ and $B(x)$ is assumed to be $B_0x^{1/2}$, which reflects the local magnitude of the magnetic field at various stations of the plate and hence preserve the validity in the vicinity of the leading edge. So, the solutions can be expanded as an asymptotic series in ξ. This series solution is valid for small ξ, that is, for forced and free convection near the leading edge of the plate, at the region where the free convection effect is smaller [9]. Hence we consider here ξ to be smaller so that we get combined effects of forced and free convection on the flow near the leading edge. Accordingly we expand the functions $f(\xi,\eta)$ and $\theta(\xi,\eta)$ in powers of ξ, that is

$$f(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \cdots \tag{12}$$

$$\theta(\xi,\eta) = \theta_0(\eta) + \xi\theta_1(\eta) + \xi^2\theta_2(\eta) + \cdots \tag{13}$$

and substituting these into equations (9)–(11) and collecting the terms up to $O(\xi^2)$ only, we get:

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$$f_0''' + \tfrac{1}{2}f_0 f_0'' = 0 \tag{14}$$

$$\theta_0'' + \tfrac{1}{2}\sigma(f_0\theta_0' - f_0'\theta_0) = 0 \tag{15}$$

$$f_0(0) = f_0'(0) = 0, \quad \theta_0(0) = -1$$
$$f_0'(\infty) = \theta_0(\infty) = 0; \tag{16}$$

$O(\xi)$

$$f_1''' + \tfrac{1}{2}f_0 f_1'' - \tfrac{3}{2}f_0' f_1' + 2f_0'' f_1 = M(f_0' - 1) - \theta_0 \tag{17}$$

$$\theta_1'' + \tfrac{1}{2}\sigma(f_0\theta_1' - 4f_0'\theta_1) = \tfrac{1}{2}\sigma(\theta_0 f_1' - 4\theta_0' f_1) \tag{18}$$

$$f_1(0) = f_1'(0) = \theta_1'(0) = 0, \quad f_1'(\infty) = \theta_1(\infty) = 0; \tag{19}$$

$O(\xi^2)$

$$f_2''' + \tfrac{1}{2}f_0 f_2'' - 3f_0' f_2' + \tfrac{7}{2}f_0'' f_2 = \tfrac{3}{2}f_1'^2 - 2f_1 f_1'' + Mf_1 - \theta_1 \tag{20}$$

$$\theta_2'' + \tfrac{1}{2}\sigma(f_0\theta_2' - 7f_0'\theta_2) = \sigma(2f_1'\theta_1 + \tfrac{1}{2}\theta_0 f_2' - 2\theta_1' f_1 - \tfrac{7}{2}\theta_0' f_2) \tag{21}$$

$$f_2(0) = f_2'(0) = \theta_2'(0) = 0, \quad f_2'(\infty) = \theta_2(\infty) = 0. \tag{22}$$

The present problem, in the absence of magnetic field, had been studied by Tingwei *et al.* [11] using the modified fourth-order Runge–Kutta method of Lapidus and Seinfeld [16]. But in the present paper we are proposing to study the problem in a different approach, the details of which are given in the following section.

3. METHOD OF SOLUTIONS

From the previous section it is evident that equations (15)–(22) are linear and may be solved independently one after another, since the first equation (14) is the well-known Blasius equation the solution of which is already known, we therefore first propose to solve equation (15). Since the function $f_0(\eta)$ is known, we may reduce the boundary value problem provided with equation (15) and the boundary condition (16) by the method of superposition [17] as given below. We now write

$$\theta_0(\eta) = \theta_{01}(\eta) + \lambda_{02}(\eta) \tag{23}$$

and substituting this into equation (15) and the boundary conditions (16) to get

$$\theta_{01}'' + \tfrac{1}{2}\sigma(f_0\theta_{01}' - f_0'\theta_{01}) = 0 \tag{24}$$

Table 1. Values of $f''_0(0)$, $f''_1(0)$, $f''_2(0)$, $\theta_0(0)$, $\theta_1(0)$ and $\theta_2(0)$ in absence of magnetic field

σ	$f''_0(0)$	$\theta_0(0)$	$f''_1(0)$	$-\theta_1(0)$	$-f''_2(0)$	$\theta_2(0)$
0.01	0.33206	8.74748†	13.0633†	17.9801†	126.949†	407.404†
		8.74754	13.0602	17.9301	126.909	407.235
0.10	0.33206	4.93984†	8.54192†	11.5080†	38.0465†	147.422†
		4.94009	6.54073	11.5086	39.0515	142.406
0.70	0.33206	2.46371†	2.48689†	2.84295†	6.24779†	14.8624†
		2.46420	2.48731	2.84355	6.24994	14.8677
1.0	0.33206	2.17879†	2.05983†	2.15080†	4.35316†	9.45458†
		2.17933	2.0603	2.15154	4.35509	9.46015
10.00	0.33206	1.00212†	0.55473†	0.30196†	0.33816†	0.38178†
		1.00234	0.55492	0.30208	0.33915	0.38216

† The values are due to Tingwei *et al.* [11].

$$\theta''_{02}+\tfrac{1}{2}\sigma(f_0\theta'_{02}-f'_0\theta_{02})=0 \tag{25}$$

$$\theta_{01}(0)=0,\quad \theta'_{01}(0)=-1,\quad \theta_{02}(0)=1,\quad \theta'_{02}(0)=0. \tag{26}$$

The initial conditions (26) are obtained on the assumptions that $\theta_0(0)=\lambda$.

Equations (24)–(26) now constitute a set of initial value problems that can be integrated without iteration by the use of any initial value solver to give θ_{01} and θ_{02}. The integrations are carried out in the domain $0\leqslant\eta\leqslant\eta_\infty$. From the above integrations knowing the values of the functions $\theta_{01}(\eta)$ and $\theta_{02}(\eta)$ at η_∞, we find the parameter λ, using the condition $\theta_0(\infty)=0$, from the following relation:

$$\lambda=-\theta_{01}(\infty)/\theta_{02}(\infty). \tag{27}$$

Finally knowing the value of λ from equation (27) one finds easily the solution for $\theta_0(\eta)$ from relation (23). During the course of integrations, for some values of σ the grid size $\Delta\eta$ as well as η_∞ had to be changed. For $0.1<\sigma<1$, $\Delta\eta$ has been taken as 0.05. The corresponding values for η_∞ were taken as 8, 10 and 12. For these three different values of η_∞ no significant changes in value of λ were observed. Again, for $\sigma\geqslant 1.0$, considering $\Delta\eta$ to be 0.02, the solutions are obtained with $\eta_\infty=10$. Finally, we have considered the value of $\Delta\eta$ to be 0.1 and the integrations were carried out with $\eta_\infty=10$, 12 and 15. For these values of η_∞ no significant change in the final value of λ had been observed, even for smaller values of σ. Therefore, in the present analysis, for $\sigma\leqslant 0.1$ we took the solutions with $\eta_\infty=12$. Following the same method all of the rest of the boundary value problems (17)–(22) are solved one after another. Table 1 compares the values of $f''_0(0)$, $\theta_0(0)$, $f''_1(0)$, $\theta_1(0)$, $f''_2(0)$ and $\theta_2(0)$ with those of Tingwei *et al.* [11] for different values of the Prandtl number σ.

Once we know the functions f_0, f_1, f_2, θ_0, θ_1 and θ_2, we may find the velocity distribution from relation (28)

$$u/U_0=f'_0(\eta)+\left(\frac{Gr}{Re^{5/2}}\right)f'_1(\eta)+\left(\frac{Gr}{Re^{5/2}}\right)f'_2(\eta)+\cdots \tag{28}$$

and the temperature distribution from the relation

$$\frac{T-T_0}{T_w-T_0}=\frac{\theta_0(\eta)+(Gr/Re^{5/2})\theta_1(\eta)+(Gr/Re^{5/2})^2\theta_2(\eta)+\cdots}{\theta_0(0)+(Gr/Re^{5/2})\theta_1(0)+(Gr/Re^{5/2})\theta_2(0)+\cdots}. \tag{29}$$

Knowing the velocity and the temperature distribution one can easily find the wall shear stress and the heat transfer rate. The wall shear stress may be expressed in terms of the local skin-friction coefficient as given below

$$c_f=\frac{2\tau_0}{\rho U_0^2}.$$

Substitution into equation (28) yields

$$\tfrac{1}{2}Re^{1/2}c_f=f''(\xi,0)$$

$$=f''_0+\left(\frac{Gr}{Re^{5/2}}\right)f''_1(0)+\left(\frac{Gr}{Re^{5/2}}\right)^2f''_2(0)+\cdots. \tag{30}$$

Finally, in terms of the Nusselt number, the heat transfer may be expressed as

$$Re^{-1/2}Nu=\frac{qx}{(T_w-T_0)k}=\left[\theta_0(0)+\left(\frac{Gr}{Re^{5/2}}\right)\theta_1(0)+\left(\frac{Gr}{Re^{5/2}}\right)^2\theta_2(0)+\cdots\right]^{-1}. \tag{31}$$

In the following section the results thus obtained from the above analysis are discussed in detail.

4. RESULTS AND DISCUSSIONS

In fact, assumptions used to establish the governing equation are particularly appropriate to liquid metals. Moreover, as liquid metals are currently used as coolants in engineering. We have pursued solutions into the lower Prandtl number ranges, e.g. 0.05 for lithium, 0.01 for mercury and 0.005 for sodium. Detailed numerical solutions having been obtained for $\sigma=10$, 1, 0.7, 0.5, 0.1, 0.05, 0.02 and 0.01. The numerical integrations of equations (14)–(22) for the above values of σ were carried out on Gould/9000 mainframe of the ICTP, Trieste, Italy.

The velocity distributions obtained from relation (28) are shown graphically in Figs. 1(a), 2(a), and 3(a). In Fig. 1(a) the curves represent the velocity profiles for different values of the magnetic field parameter when the fluid is air and for the value of the buoyancy parameter $\xi=0.01$. From this figure it is observed that the presence as well as the increase in the magnetic field leads to decrease in the velocity field, that is, it retards the flow field. From Figs. 2(a) and 3(a) we further observe flow separation or reverse type of flow for smaller values of the Prandtl number σ (0.05 and 0.01 which represent lithium and mercury, respectively), which is unacceptable physically, since the magnetic field can only return and not produce flow in the reverse direction [18, 19]. But from Fig. 3(a) it is observed that this flow condition can be improved by a smaller increment in the buoyancy force in the flow field.

Figures 1(b), 2(b) and 3(b) are representing the temperature distributions in the flow field. In Figs. 1(b) and 2(b) we see that the presence as well as increase in the magnetic field leads to a rise in the temperature distribution in the flow field. This further increases owing to a rise in the buoyancy force parameter (Fig. 3(b)).

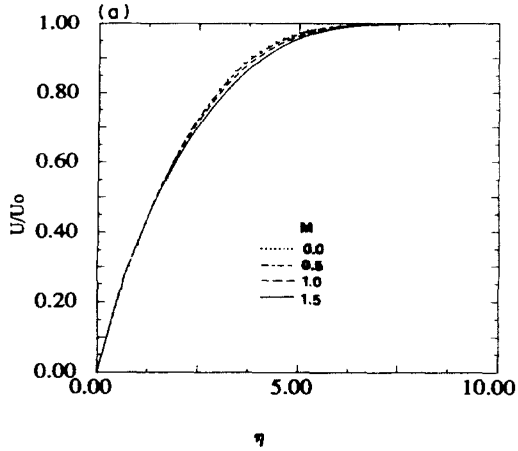

FIG. 1(a). Velocity profiles against η for $\sigma = 0.7$, $\xi = 0.1$.

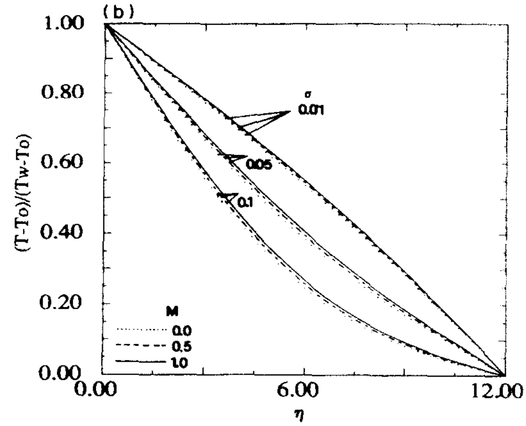

FIG. 2(b). Temperature profiles against η for $M = 1.0$ and for different values of σ and ξ.

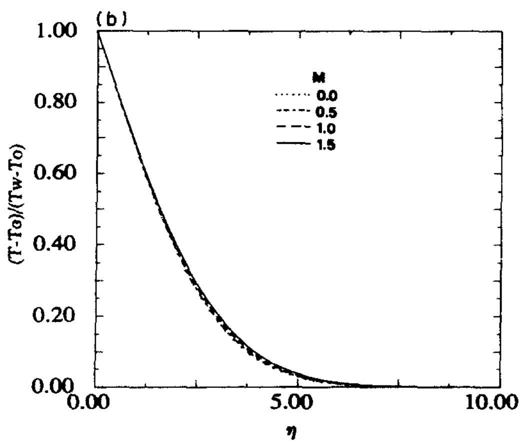

FIG. 1(b). Temperature profiles against η for $\sigma = 0.7$, $\xi = 0.1$.

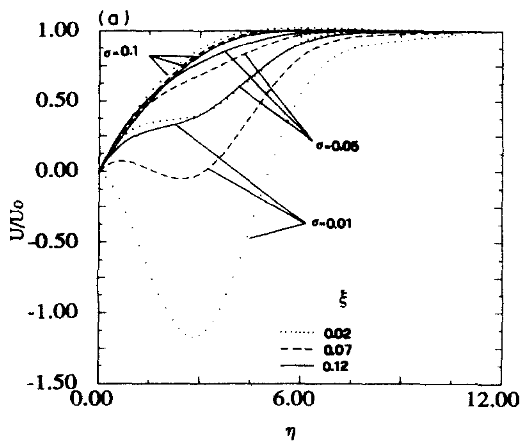

FIG. 3(a). Velocity profiles against η for $\xi = 0.1$ and for different values of σ and M.

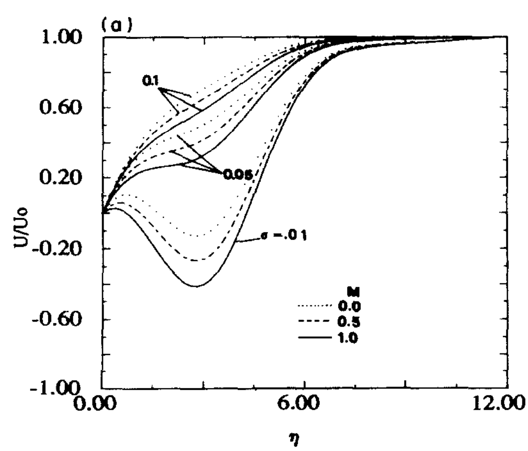

FIG. 2(a). Velocity profiles against η for $M = 1.0$ and for different values of σ and ξ.

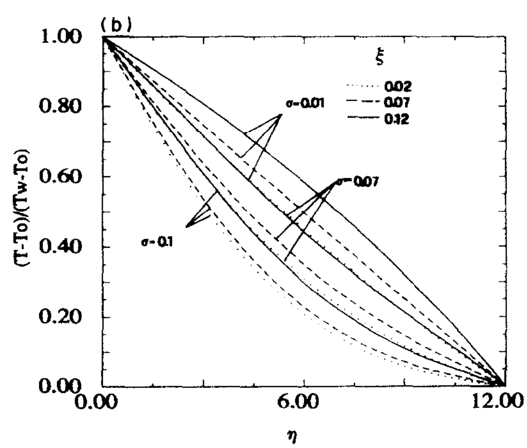

FIG. 3(b). Temperature profiles against η for $\xi = 0.1$ and for different values of σ and M.

Table 2. Values of local skin-friction coefficient and the Nusselt number for $\xi = 0.1$

		$2\sqrt{(Re)}\,c_f$			$-Re^{1/2}\,Nu$		
M	σ	0.01	0.1	0.7	0.01	0.1	0.7
0.0		0.36389	0.59561	0.51829	0.09066	0.19181	0.42945
0.5		0.29276	0.57852	0.54226	0.08832	0.18610	0.42729
1.0		0.21064	0.55553	0.56035	0.08598	0.18012	0.42240
2.0		0.2878	0.49190	0.57885	0.08141	0.16766	0.40523

In Table 1 we have entered the values of $f_0''(0)$, $f_1''(0)$, $f_2''(0)$, $\theta_0(0)$, $\theta_1(0)$ and $\theta_2(0)$ for different values of the Prandtl number in the absence of the magnetic field for comparison with the results of Tingwei *et al.* [11]. It can easily be seen that the largest difference between the present values and the corresponding values of Tingwei *et al.* are less than 2%. The value of $f_1''(0)$ for $\sigma = 0.1$ obtained by Tingwei *et al.* which is 8.54192, must be a misprint, since the more accurate one is 6.54192. Finally Table 2 represents the values of the wall shear stress and the rate of heat transfer in terms of skin-friction coefficient and Nusselt number, respectively, for magnetic field parameter $M = 0.0, 0.5, 1.0, 2$ and for Prandtl number $\sigma = 0.01, 0.1$ and 0.7 at $\xi = 0.1$. From this table one may conclude that the value of the skin friction decreases whereas that of the heat transfer increases owing to increase in the magnetic field strength.

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REFERENCES

1. E. M. Sparrow, R. Eichorn and J. L. Gregg, Combined forced and free convection in a boundary layer flow, *Physics Fluids* **2**, 319–328 (1959).
2. E. M. Sparrow and J. L. Gregg, Buoyancy effects in forced convection flow and heat transfer, *ASME J. Appl. Mech.* **83**, 133–134 (1959).
3. J. H. Merkin, The effect of buoyancy forces on the boundary layer flow over semi-infinite vertical flat plate in a uniform free stream, *J. Fluid Mech.* **35**, 439–450 (1969).
4. J. R. Lloyd and E. M. Sparrow, Combined forced and free convection flow on vertical surfaces, *Int. J. Heat Mass Transfer* **13**, 434–438 (1970).
5. G. Wilks, Combined forced and free convection flow on vertical surfaces, *Int. J. Heat Mass Transfer* **16**, 1958–1964 (1973).
6. L. G. Leal, Combined forced and free convection heat transfer from a horizontal flat plate, *ZAMP* **24**, 20–42 (1973).
7. T. S. Chen, E. M. Sparrow and A. Mucoglu, Mixed convection in boundary layer flow on a horizontal plate, *ASME J. Heat Transfer* **99**, 66–71 (1977).
8. A. Mucoglu and T. S. Chen, Mixed convection on inclined surfaces, *ASME J. Heat Transfer* **101**, 422–426 (1979).
9. L. S. Yao, Two-dimensional mixed convection along a flat plate, *ASME J. Heat Transfer* **109**, 440–445 (1987).
10. M. S. Rajn, X. R. Liu and C. K. Law, A formulation of combined forced and free convection past horizontal and vertical surfaces, *Int. J. Heat Mass Transfer* **27**, 2215–2224 (1984).
11. G. Tingwei, R. Bachrun and M. Daguent, Influence de la convection naturel le sur la convection forcee an-dessus d'une surface plane verticale voumise a un flux de rayonnement, *Int. J. Heat Mass Transfer* **25**, 1061–1065 (1982).
12. P. S. Lykoudis, On a class of magnetic seminar boundary layers, *Proc. Heat Transfer Fluid Mech. Inst.* 176–184 (1959).
13. A. S. Gupta, Steady and transient free convection of an electrically conducting fluid from a vertical plate in the presence of a magnetic field, *Appl. Scient. Res.* **9A**, 319–333 (1961).
14. M. H. Cobble, Free convection with mass transfer under the influence of a magnetic field, *Nonlinear Analysis* **3**, 135–143 (1979).
15. G. Wilks and R. Hunt, Magnetohydrodynamic free convection flow about a semi-infinite plate at whose surface the heat flux is uniform, *ZAMP* **35**, 34–49 (1984).
16. L. Lapidus and J. H. Seinfeld, *Numerical Solution of Ordinary Differential Equation*. Academic Press, New York (1971).
17. T. Y. Na, *Computational Methods in Engineering Boundary Value Problems*. Academic Press, New York (1979).
18. V. J. Rossow, On flow of electrically conducting fluid over a flat plate in the presence of a transverse magnetic field, NACA Rept. 1358 (1958).
19. R. D. Hugelman, An MHD boundary layer compatibility condition, *AIAA J.* **3**, 1367–1368 (1965).